The Equation of State and Quark Number Susceptibility in Hard-Dense-Loop Approximation

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Abstract

Based on the method proposed in [H. S. Zong, W. M. Sun, Phys. Rev. **D 78**, 054001 (2008)], we calculate the equation of state (EOS) of QCD at zero temperature and finite quark chemical potential under the hard-dense-loop (HDL) approximation. A comparison between the EOS under HDL approximation and the cold, perturbative EOS of QCD proposed by Fraga, Pisarski and Schaffner-Bielich is made. It is found that the pressure under HDL approximation is generally smaller than the perturbative result. In addition, we also calculate the quark number susceptibility (QNS) at finite temperature and finite chemical potential under hard-thermal/dense-loop (HTL/HDL) approximation and compare our results with the corresponding ones in the previous literature.

The investigation of the equation of state (EOS) for cold and dense strongly interacting matter and its consequence for the possible phases of quantum chromodynamics (QCD) plays a crucial role in the study of neutron stars in astrophysics [1, 2]. At present lattice QCD provides the most powerful tool in studying QCD thermodynamics. While the Monte Carlo simulation works well and has achieved great success in dealing with finite temperature QCD thermodynamics, it encounters the notorious sign problem in the situation of finite quark chemical potential and cannot be applied [3]. To circumvent the sign problem, several methods have been proposed. Among them are the reweighting technique [4], analytic continuation from imaginary chemical potential [5, 6] and Taylor expansion in μ around $\mu = 0$ [7–9]. Although these methods have achieved some success in treating finite μ physics, they are far from being complete and even suffer from some limitations and drawbacks [10]. So we expect that continuum model studies should be complementary to the lattice simulations in the exploration of QCD thermodynamics at finite μ. At high enough temperature (T) and/or the quark chemical potential (μ), the hard-thermal/dense-loop (HTL/HDL) approximation is thought to be a good approximation for QCD [11]. In the present paper, based on the method proposed in Ref. [12], we try to calculate the EOS of QCD at zero temperature and high enough quark chemical potential under HDL approximation. Since the HDL approximation is a good approximation for QCD at high enough quark chemical potential, the EOS derived with the aid of it should have a relatively good QCD foundation and one expects that it can be applied to the study of neutron star. In addition, with the method proposed in the present paper we can calculate another important quantity for the study of QCD phase transition, the quark number susceptibility (QNS), in the HTL/HDL approximation. The QNS measures the intrinsic statistical fluctuations in a system close to thermal equilibrium and is thought to play an important role in identifying the critical end point in the QCD phase diagram [13–20].

Let us start with the renormalized partition function of QCD at zero T and finite μ which reads

$$\mathcal{Z}[\mu] = \int \mathcal{D}\bar{q}_R \mathcal{D}q_R \mathcal{D}A_R \, \exp\left\{-S_R[\bar{q}_R, q_R, A_R] + \int d^4x \, \mu Z_2 \bar{q}_R(x)\gamma_4 q_R(x)\right\}, \qquad (1)$$

where $S_R[\bar{q}_R, q_R, A_R]$ is the standard renormalized Euclidean QCD action with q_R being the renormalized quark fields with three flavors and three colors, $Z_2 = Z_2(\zeta^2, \Lambda^2)$ is the quark wave-function renormalization constant (ζ is the renormalization point and Λ is the

regularization mass-scale). Here we leave the ghost field term and its integration measure to be understood.

The pressure density $\mathcal{P}(\mu)$ is given by

$$\mathcal{P}(\mu) = \frac{1}{\mathcal{V}} \ln \mathcal{Z}[\mu], \tag{2}$$

where $\mathcal{V}$ is the four-volume normalizing factor. The above expression for $\mathcal{P}(\mu)$ is just the EOS. From the pressure density one immediately obtains the quark number density:

$$\rho(\mu) = \frac{\partial \mathcal{P}(\mu)}{\partial \mu}. \tag{3}$$

According to Ref. [12], one has the following well-known result

$$\rho(\mu) = -\frac{Z_2}{\mathcal{V}} \mathrm{Tr} \left\{ G_R[\mu] \gamma_4 \right\} = -N_c N_f Z_2 \int \frac{d^4 p}{(2\pi)^4} \mathrm{tr} \left\{ G_R[\mu](p) \gamma_4 \right\}, \tag{4}$$

where $G_R[\mu](p)$ is the renormalized quark propagator at finite chemical potential and N_c, N_f denote the number of colors and flavors, respectively. Here, Tr denotes trace operation over color, flavor, Dirac and coordinate indices, while tr denotes trace operation over Dirac indices only.

Integrating both sides of the equation $\rho(\mu) = \partial \mathcal{P}(\mu)/\partial \mu$ gives

$$\mathcal{P}(\mu) = \mathcal{P}(\mu)|_{\mu=0} + \int_0^\mu d\mu' \rho(\mu') = \mathcal{P}(\mu)|_{\mu=0} - N_c N_f Z_2 \int_0^\mu d\mu' \int \frac{d^4 p}{(2\pi)^4} \mathrm{tr} \left\{ G_R[\mu'](p) \gamma_4 \right\}, \tag{5}$$

where $\mathcal{P}(\mu)|_{\mu=0}$ is an integration constant which represents the pressure density at $\mu = 0$. Here one should note that $\mathcal{P}(\mu)|_{\mu=0}$ is a constant independent of μ and the whole nontrivial μ dependence of $\mathcal{P}(\mu)$ is contained in the integration term which is totally determined by the renormalized quark propagator $G_R[\mu](p)$. Formally Eq. (5) provides a model-independent formula for calculating the pressure density at finite μ. However, at present we still do not know how to calculate the quark propagator at finite chemical potential from first principles of QCD. So, when one uses Eq. (5) to calculate the pressure density, one has to resort to various QCD models or approximations. In this paper we will use the HDL approximation.

The quark propagator under HTL/HDL approximation can be written as [11]

$$G_R(p) = -Z_2^{-1} \frac{1}{D_+(p)} \frac{\gamma_4 + i\hat{\mathbf{p}} \cdot \vec{\gamma}}{2} - Z_2^{-1} \frac{1}{D_-(p)} \frac{\gamma_4 - i\hat{\mathbf{p}} \cdot \vec{\gamma}}{2}, \tag{6}$$

where $\hat{\mathbf{p}} = \vec{p}/|\vec{p}|$ and $p_4 = (2n+1)\pi T$ ($n \in \mathbf{Z}$) are the fermion Matsubara frequencies. The form of the functions $D_\pm(p)$ is

$$D_\pm(p) = -ip_4 + \mu \pm |\vec{p}| + \frac{m_q^2}{|\vec{p}|} \left[Q_0 \left(\frac{ip_4 - \mu}{|\vec{p}|} \right) \mp Q_1 \left(\frac{ip_4 - \mu}{|\vec{p}|} \right) \right], \tag{7}$$

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where $m_q \equiv g\sqrt{(T^2 + \mu^2/\pi^2)/6}$ is the quark thermal mass with g being the strong coupling constant, Q_0 and Q_1 are Legendre functions of the second kind. Substituting Eq. (6) into Eq. (4) gives

$$\rho(\mu) = 2N_c N_f \int \frac{d^3\vec{p}}{(2\pi)^3} T \sum_n \left[\frac{1}{D_+} + \frac{1}{D_-} \right]. \tag{8}$$

According to a familiar method in thermal field theory, the frequency sum in Eq. (8) can be calculated by means of contour integral

$$T\sum_n \left[\frac{1}{D_+} + \frac{1}{D_-} \right] + \int_{-i|\vec{p}|-i\mu}^{i|\vec{p}|-i\mu} \frac{dp_4}{2\pi} \mathrm{Disc} \left[\frac{1}{D_+} + \frac{1}{D_-} \right] \frac{1}{2} \tanh \frac{ip_4}{2T}$$
$$= \int_{C_1 \cup C_2} \frac{dp_4}{2\pi} \left[\frac{1}{D_+} + \frac{1}{D_-} \right] \frac{1}{2} \tanh \frac{ip_4}{2T}. \tag{9}$$

The integral contour is shown in Fig. 1. Now let us calculate the integral. Defining

$$f(p_4) := \left[\frac{1}{D_+} + \frac{1}{D_-} \right] \frac{1}{2} \tanh \frac{ip_4}{2T} \tag{10}$$

and closing C_1 and C_2 with large half circles, one obtains the following

$$\int_{C_1 \cup C_2} \frac{dp_4}{2\pi} \left[\frac{1}{D_+} + \frac{1}{D_-} \right] \frac{1}{2} \tanh \frac{ip_4}{2T} = \frac{1}{2\pi}(-2\pi i) \sum_j \mathrm{Res}\{f(\zeta_j)\}, \tag{11}$$

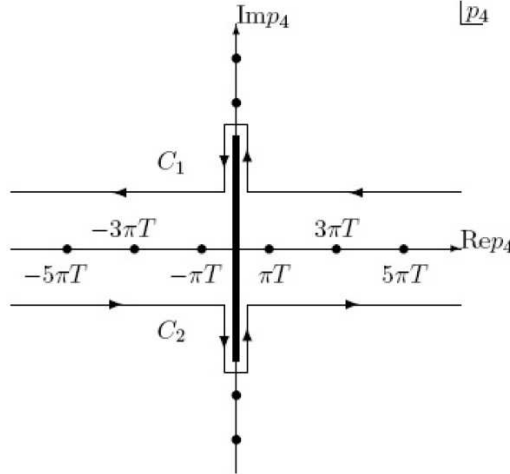where ζ_j are the poles of $f(p_4)$ located in the imaginary axis. The poles of $1/D_+$ are

FIG. 1: The integral contour

$$\zeta_1 = -i(\omega_+ + \mu), \quad \mathrm{Res}\left\{ \frac{1}{D_+(\zeta_1)} \right\} = iZ_+ \tag{12}$$

$$\zeta_2 = i(\omega_- - \mu), \quad \mathrm{Res}\left\{ \frac{1}{D_+(\zeta_2)} \right\} = iZ_- \tag{13}$$

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and the poles of $1/D_-$ are

$$\zeta_3 = i(\omega_+ - \mu), \quad \text{Res}\left\{\frac{1}{D_-(\zeta_3)}\right\} = iZ_+ \tag{14}$$

$$\zeta_4 = -i(\omega_- + \mu), \quad \text{Res}\left\{\frac{1}{D_-(\zeta_4)}\right\} = iZ_-, \tag{15}$$

where $\omega_\pm(|\vec{p}|)(> |\vec{p}|)$ are the solutions of the equations

$$\frac{|\vec{p}|(\omega_+ - |\vec{p}|)}{m_q^2} - 1 = \frac{1}{2}\left(1 - \frac{\omega_+}{|\vec{p}|}\right)\ln\frac{\omega_+ + |\vec{p}|}{\omega_+ - |\vec{p}|}, \tag{16}$$

$$\frac{|\vec{p}|(\omega_- + |\vec{p}|)}{m_q^2} + 1 = \frac{1}{2}\left(1 + \frac{\omega_-}{|\vec{p}|}\right)\ln\frac{\omega_- + |\vec{p}|}{\omega_- - |\vec{p}|} \tag{17}$$

and $Z_\pm(|\vec{p}|)$ are

$$Z_\pm(|\vec{p}|) = \frac{\omega_\pm - |\vec{p}|^2}{2m_q^2}. \tag{18}$$

From these results one can calculate the right hand side of Eq. (11):

$$\int_{C_1 \cup C_2} \frac{dp_4}{2\pi}\left[\frac{1}{D_+} + \frac{1}{D_-}\right]\frac{1}{2}\tanh\frac{ip_4}{2T} = \frac{1}{2\pi}(-2\pi i)\sum_j \text{Res}\{f(\zeta_j)\} \tag{19}$$

$$= -\frac{i}{2}\left[iZ_+\tanh\frac{\omega_+ + \mu}{2T} + iZ_-\tanh\frac{-(\omega_- - \mu)}{2T} + iZ_+\tanh\frac{-(\omega_+ - \mu)}{2T} + iZ_-\tanh\frac{\omega_- + \mu}{2T}\right].$$

The second term in the left hand side of Eq. (9) can be expressed as

$$\int_{-i|\vec{p}|-i\mu}^{i|\vec{p}|-i\mu} \frac{dp_4}{2\pi}\text{Disc}\left(\frac{1}{D_+} + \frac{1}{D_-}\right)\frac{1}{2}\tanh\frac{ip_4}{2T} = \int_{-|\vec{p}|}^{|\vec{p}|} \frac{d\omega}{2\pi}[\rho_+(\omega) + \rho_-(\omega)]\frac{1}{2}\tanh\frac{\omega - \mu}{2T}, \tag{20}$$

where $\rho_\pm(\omega, |\vec{p}|)$ are the familiar spectral functions of the quark propagator $(x \equiv \omega/|\vec{p}|)$ [11]:

$$\rho_\pm(\omega, |\vec{p}|) = 2\pi[Z_\pm(|\vec{p}|)\delta(\omega - \omega_\pm(|\vec{p}|)) + Z_\mp(|\vec{p}|)\delta(\omega + \omega_\mp(|\vec{p}|))]$$
$$+\frac{\pi m_q^2(1 \mp x)\theta(1 - x^2)}{|\vec{p}|^3}\left\{\left[1 \mp x + \frac{m_q^2}{|\vec{p}|^2} \pm \frac{m_q^2}{2|\vec{p}|^2}(1 \mp x)\ln\frac{1+x}{1-x}\right]^2\right.$$
$$\left.+\frac{\pi^2 m_q^4}{4|\vec{p}|^4}(1 \mp x)^2\right\}^{-1}. \tag{21}$$

Combining Eqs. (19), (20) and (9), one can obtain

$$T\sum_n\left[\frac{1}{D_+} + \frac{1}{D_-}\right] = Z_+\left[n_F(\omega_+ - \mu) - n_F(\omega_+ + \mu)\right] + Z_-\left[n_F(\omega_- + \mu) - n_F(\omega_- - \mu)\right]$$
$$-\frac{1}{2}\int_{-|\vec{p}|}^{|\vec{p}|} \frac{d\omega}{2\pi}[\rho_+(\omega) + \rho_-(\omega)][1 - 2n_F(\omega - \mu)], \tag{22}$$

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where $n_F(\omega)$ is the Fermi distribution function

$$n_F(\omega) \;=\; \frac{1}{\exp(\omega/T)+1}. \tag{23}$$

Therefore the quark number density at finite μ and T is

$$\begin{aligned}
\rho(\mu,T) \;=\;& 2N_cN_f \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ Z_+ \Big[n_F(\omega_+ - \mu) - n_F(\omega_+ + \mu) \Big] \right.\\
&\left. + Z_- \Big[n_F(\omega_- + \mu) - n_F(\omega_- - \mu) \Big] \right\} \\
&- N_cN_f \int \frac{d^3\vec{p}}{(2\pi)^3} \int_{-|\vec{p}|}^{|\vec{p}|} \frac{d\omega}{2\pi} [\rho_+(\omega) + \rho_-(\omega)][1 - 2n_F(\omega - \mu)].
\end{aligned} \tag{24}$$

From Eq. (24) one can clearly see that the quark number density contains two terms, one is the contribution from the quasi-particle poles and the other is the contribution from the Landau damping. In the limit of $T \to \infty$, ρ tends to the free quark gas result ($Z_+ \to 1$, $Z_- \to 0$ and the integral is dominated by $p \sim \omega \sim T$ region).

Before proceeding the calculation of EOS, let us turn to the calculation of the QNS, which is defined as

$$\chi(\mu,T) \;\equiv\; \frac{\partial \rho(\mu,T)}{\partial \mu}. \tag{25}$$

As mentioned before, the QNS plays an important role in identifying the critical end point in the QCD phase diagram. There are many calculations of QNS under HTL approximation in the literature [21–25]. Generally, in order to calculate the QNS, one has to calculate the quark propagator and the quark-meson vertex in the vector meson channel at zero total momentum separately. In Ref. [25], using the fact that $\partial/\partial\mu$ can be replaced by $\partial/\partial(-ip_4)$ and with the help of the Ward-Takahashi identity, the authors calculate the QNS under HTL approximation only through the HTL quark propagator. Here a question arises: is the information of μ dependence lost by such a method in Ref. [25]? Now we will answer this question in a straightforward way by calculating the QNS directly from Eqs. (24) and (25). Because $Z_\pm$, $\rho_\pm$ and $\omega_\pm$ depend on μ only through m_q, one would obtain the following

$$\chi(\mu,T) \;=\; \frac{\partial \rho(\mu,T,m_q)}{\partial \mu} + \frac{\partial \rho(\mu,T,m_q)}{\partial m_q}\frac{\partial m_q}{\partial \mu}. \tag{26}$$

Setting $\mu = 0$ in the above equation gives

$$\chi(\mu = 0,T) \;=\; \left. \frac{\partial \rho(\mu,T,m_q)}{\partial \mu} \right|_{\mu=0} + \left[\frac{\partial \rho(\mu,T,m_q)}{\partial m_q}\frac{\partial m_q}{\partial \mu} \right]_{\mu=0}. \tag{27}$$

One can easily find that the second term in Eq. (27) is zero $((\partial m_q/\partial\mu)_{\mu=0} = 0)$ and the QNS at finite T and zero μ is

$$
\begin{aligned}
\chi(\mu = 0, T) \;=\; & \frac{4N_cN_f}{T}\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\frac{\omega_+^2 - \vec{p}^2}{2m_q^2}n_F(\omega_+)(1 - n_F(\omega_+)) + \frac{\omega_-^2 - \vec{p}^2}{2m_q^2}n_F(\omega_-)(1 - n_F(\omega_-))\right]_{\mu=0} \\
& + \frac{2N_cN_f}{T}\int\frac{d^3\vec{p}}{(2\pi)^3}\int_{-|\vec{p}|}^{|\vec{p}|}\frac{d\omega}{2\pi}\Big\{[\rho_+(\omega) + \rho_-(\omega)]n_F(\omega)(1 - n_F(\omega))\Big\}_{\mu=0}.
\end{aligned}
\tag{28}
$$

This is the same as the result in Ref. [25], as one expected in advance. Here it should be stressed that the method used to calculate the QNS in the present paper is different from the one used in Ref. [25] in which the Ward-Takahashi identity is adopted to avoid the differentiation over μ. The result obtained here can be regarded as a self-check of the calculation in Ref. [25]. As is pointed out in Ref. [25], the QNS contains the contributions from both the Landau damping and the quasi-particle poles. This is different from the result in Ref. [23] (see Ref. [25] for more detail).

After the discussion of QNS let us continue the calculation of EOS. The zero temperature result of ρ reads

$$
\begin{aligned}
\rho(\mu) \;=\; & 2N_cN_f\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\theta(\mu - \omega_+)Z_+ - \theta(\mu - \omega_-)Z_-\right] \\
& + N_cN_f\int\frac{d^3\vec{p}}{(2\pi)^3}\int_{-|\vec{p}|}^{|\vec{p}|}\frac{d\omega}{2\pi}[\rho_+(\omega) + \rho_-(\omega)][1 - 2\theta(\omega - \mu)].
\end{aligned}
\tag{29}
$$

It should be pointed out that according to Eq. (4), in order to calculate the quark number density one needs to know the quark propagator in the whole momentum range. However, the HTL/HDL approximation is only valid for external momentum much smaller than T or μ. Therefore, as in Ref. [25], for a consistent calculation of the quark number density under the HDL approximation one should introduce a momentum cutoff Λ_{HDL} below which the HDL quark propagator (6) is applicable. In our calculation we choose $\Lambda_{HDL} = g\mu/\pi$. We make this choice because $g\mu/\pi$ is an important energy scale to identify the soft momentum in HDL approximation (for HTL this would be gT, see Ref. [25]). If we limit the range of integration of the contribution of Landau damping in Eq. (29) to the region $|\vec{p}| \le g\mu/\pi$, we will get the numerical result of the quark number density shown in Fig. 2 ($N_c = 3$, $N_f = 2$). From Fig. 2 it can be seen that the quark number density under HDL approximation is slightly smaller than that of free quark gas ($\rho_{free}(\mu) = N_cN_f\mu^3/3\pi^2$), which indicates that the attractive interaction dominate in the high density case. Here we want to stress that the result in Fig. 2 has minor sensitivity to the variation of the cut-off $g\mu/\pi$. For example, if one

sets the cut-off to be $2g\mu/\pi$, one finds that the change of ρ is about 10%. This is because the quark number density (and susceptibility) is determined essentially by the quasi-particle contribution within the given approximations.
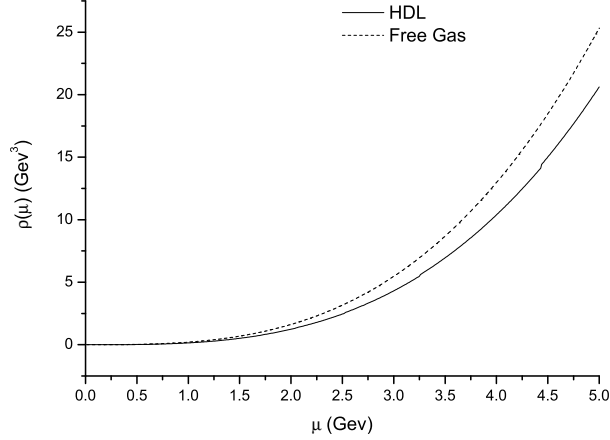

FIG. 2: The quark number density under HDL approximation

Now we can calculate the EOS under HDL approximation (here we neglect the constant term $\mathcal{P}(\mu)|_{\mu=0}$ because when applying our EOS to the study of neutron star, owing to the boundary condition imposed on the surface of neutron star, the constant term $\mathcal{P}(\mu)|_{\mu=0}$ does not contribute to the mass-radius relation)

$$\mathcal{P}(\mu) = 2N_cN_f \int_0^\mu d\mu' \int \frac{d^3\vec{p}}{(2\pi)^3} \left[\theta(\mu' - \omega_+)Z_+ - \theta(\mu' - \omega_-)Z_-\right]$$
$$+ N_cN_f \int_0^\mu d\mu' \int \frac{d^3\vec{p}}{(2\pi)^3} \int_{-|\vec{p}|}^{|\vec{p}|} \frac{d\omega}{2\pi} [\rho_+(\omega) + \rho_-(\omega)][1 - 2\theta(\omega - \mu')]. \qquad (30)$$

The numerical results are shown in Fig. 3 with $P_{free} = N_cN_f\mu^4/(12\pi^2)$ being the pressure density of the free quark gas. It can be seen that the pressure density under the HDL approximation is smaller than that of the free quark gas.

So far we have derived an expression of the nontrivial μ dependence of the pressure density in the framework of the HDL approximation. Now let us give a discussion of the features of our approach. The study of EOS of quark matter is a long-standing subject of QCD study. Just as was mentioned in the introduction, although lattice QCD has achieved some success in treating small μ physics, they are far from being complete in the case of large chemical potential problem. So one expects that continuum model studies should be

complementary to the lattice simulations in the study of QCD thermodynamics at finite μ. When the chemical potential is large enough, one may naively expects that perturbative calculation is feasible because of asymptotic freedom. However, as was pointed out in Ref. [26], the perturbative calculation of partition function of QCD at finite T or μ encounters the convergence problem and one has to resort to resummation techniques. From the point of view of functional integral approach, the calculation of partition function of QCD amounts to the calculation of all possible vacuum bubble diagrams. This is a hard work. In order to avoid the problem of calculating the complicated vacuum bubble diagrams, in our work we adopt the approach proposed in Ref. [12], in which the nontrivial μ dependence of the pressure density is totally determined by the full quark propagator at finite μ (the quark propagator is the simplest Green function of QCD). If the input dressed quark propagator at finite μ is exact, the pressure density calculated with the aid of it will also be exact. At present it is not possible to determine reliably the dressed quark propagator at finite μ from first principles of QCD. So one has to resort to various models inspired by QCD. It is generally believed that the HTL/HDL approximations are good approximations of QCD when T or μ is large enough. So in the present work we employ the HDL approximation to calculate the EOS of quark matter. Compared with other approaches of calculating the EOS of quark matter in the literature, our approach is simpler and has a better QCD foundation.

The behavior of the pressure density under HDL approximation shown in Fig. 3 is qualitatively differs from that of the naive perturbation theory. For example, in Ref. [27] Fraga, Pisarski and Schaffner-Bielich (FPS) proposed the following cold, perturbative EOS of QCD

$$\mathcal{P}_{FPS}(\mu) = \frac{N_f \mu^4}{4\pi^2}\left\{1 - 2\left(\frac{\alpha_s}{\pi}\right) - \left[G + N_f \ln\frac{\alpha_s}{\pi} + \left(11 - \frac{2}{3}N_f\right)\ln\frac{\bar{\Lambda}}{\mu}\right]\left(\frac{\alpha_s}{\pi}\right)^2\right\}, \qquad (31)$$

where $G = G_0 - 0.536N_f + N_f \ln N_f$, $G_0 = 10.374 \pm 0.13$ and $\bar{\Lambda}$ is the renormalization subtraction point. The scale dependence of the strong coupling constant $\alpha_s(\bar{\Lambda})$ is taken as

$$\alpha_s(\bar{\Lambda}) = \frac{4\pi}{\beta_0 u}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln(u)}{u} + \frac{4\beta_1^2}{\beta_0^4 u^2}\left(\left(\ln(u) - \frac{1}{2}\right)^2 + \frac{\beta_2\beta_0}{8\beta_1^2} - \frac{5}{4}\right)\right],$$

where $u = \ln(\bar{\Lambda}^2/\Lambda_{\overline{MS}}^2)$, $\beta_0 = 11 - 2N_f/3$, $\beta_1 = 51 - 19N_f/3$, and $\beta_2 = 2857 - 5033N_f/9 + 325N_f^2/27$. For $N_f = 3$, $\Lambda_{\overline{MS}} = 365$ MeV. The only freedom in the model of Ref. [27] is the choice of the ratio $\bar{\Lambda}/\mu$, which is taken to be 2 in that reference. The comparison between the HDL result and FPS result is shown in Fig. 3. From Fig. 3 it can be seen that when μ is

smaller than about 1 GeV, the HDL pressure density is smaller than the perturbative result to the α_s order, but is larger than that to the α_s^2 order. When μ is larger than about 1 GeV, the HDL pressure density is smaller than the FPS one, and an important result is that when μ tends to infinity, the HDL pressure density tends to the free quark gas result much more slowly than does the FPS one. As a comparison, the EOS obtained under Dyson-Schwinger equations (DSEs) approach [12] with different parameters are also shown in Fig. 3.
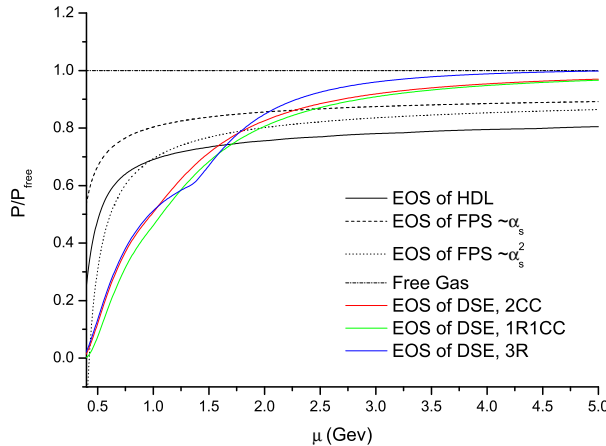

FIG. 3: The different EOS

It should be noted that in drawing the EOSs shown in Fig. 3, we have neglected the term $\mathcal{P}(\mu)|_{\mu=0}$. This does not mean that the term $\mathcal{P}(\mu)|_{\mu=0}$ is unimportant. In fact, it is an important quantity. It enters the energy density, which is relevant for integrating the Tolman-Oppenheimer-Volkoff equations. At present it is not possible to calculate reliably $\mathcal{P}(\mu)|_{\mu=0}$ from first principles of QCD. One can only calculate it using various models inspired by QCD, for example, using the CJT effective action (see, for instance, Ref. [28]). It is obvious that the existing calculations of $\mathcal{P}(\mu)|_{\mu=0}$ in the literatures are all model dependent. In our work, we have not considered the term $\mathcal{P}(\mu)|_{\mu=0}$. The main reason is that this term does not affect the relative relation of different EOSs displayed in Fig. 3, since all of the curves shown in Fig. 3 do not contain this term. If one tries to add the term $\mathcal{P}(\mu)|_{\mu=0}$ in the EOS and then make comparison between different EOSs, one has to add it in all different EOSs. Therefore the main conclusion of the present paper does not change.

Now it is time to discuss the range of applicability of the EOS presented in this paper. For the low temperature and high chemical potential regime, our knowledge is quite limited.

Experimentally, it is impossible to achieve such a condition in laboratories on earth. A natural laboratory holding such cold, highly compressed matter is a quark star. So the only obvious relevance of the presented EOS is for quark stars. It is well known that in astrophysics the study of quark star depends crucially on the assumed EOS [1, 2]. Therefore it is interesting to apply the present EOS to the study of quark star. In our future work [29] we shall consider the structure of the quark star for the three flavor situation, where we shall study the mass-radius relation, mass-central-density relation, and distribution of inner mass and chemical potential for this kind of quark star. In addition, we shall compare the results calculated from the present EOS with those calculated from other EOSs in Fig. 3.

To summarize, in this paper we calculate the EOS of QCD at zero temperature and finite quark chemical potential under the HDL approximation. It is found that when μ tends to infinity the HDL pressure density tends to the free quark gas result much more slowly than that of the cold, perturbative EOS of QCD of Fraga, Pisarski and Schaffner-Bielich. We also give the expression of QNS at finite T and μ. It is found that when $\mu \to 0$, the result of QNS is the same as the one obtained in Ref. [25].

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